Lecture 8

Compression beyond iid data

Recap

- Huffman, Arithmetic, ANS
- We know how to achieve the entropy in a computationally efficient manner.

```
$ cat sherlock.txt
    ...
      In mere size and strength it was a terrible creature which was
      lying stretched before us. It was not a pure bloodhound and it
      was not a pure mastiff; but it appeared to be a combination of
      the two—gaunt, savage, and as large as a small lioness. Even now
      in the stillness of death, the huge jaws seemed to be dripping
      with a bluish flame and the small, deep-set, cruel eyes were
      ringed with fire. I placed my hand upon the glowing muzzle, and
      as I held them up my own fingers smouldered and gleamed in the
      darkness.

      “Phosphorus,” I said.

      “A cunning preparation of it,” said Holmes, sniffing at the dead
    ...
```

Let's try and compress this 387 KB book.

```
>>> from core.data_block import DataBlock
>>>
>>> with open("sherlock.txt") as f:
>>>     data = f.read()
>>>
>>> print(DataBlock(data).get_entropy()*len(data)/8, "bytes")

199833 bytes
```

```
$ gzip < sherlock.txt | wc -c
134718

$ bzip2 < sherlock.txt | wc -c
99679
```

What's up? What are we missing here? Any suggestions?

1. Data is not iid.
2. Maybe the entire file doesn't have the same distribution (think concatenating an English novel with a Hindi novel).

In the next few lectures, we will discuss methods to compress real-life data, attempting to handle non-iid data whose distribution we do not know a priori.

Beyond iid data

- text
- images
- video
- tables
- basically anything in real life

Probability recap

Recall for $U^n = (U_1, \ldots, U_n)$:

for iid

$$P(U^n) = \Pi_{i=1}^n P(U_i)$$

in general

$$P(U^n) = \Pi_{i=1}^n P(U_i|U^{i-1}) = \Pi_{i=1}^n P(U_i|U_1, \ldots, U_{i-1})$$

Stochastic process (aka random process)

Given alphabet $\mathcal{U}$, a *stochastic process* $(U_1, U_2, \dots)$ can have arbitrary dependence across the elements and is characterized by:

$P((U_1, U_2, \dots, U_n) = (u_1, u_2, \dots, u_n))$ for $n = 1, 2, \dots$ and $(u_1, u_2, \dots, u_n) \in \mathcal{U}^n$.

Way too general to be of much use.

Stationary stochastic process

> **Definition: Stationary Process**

> A stationary process is a stochastic process that is time-invariant, i.e., the probability distribution doesn't change with time (here time refers to the index in the sequence). More precisely, we have
>
> $$P(U_1 = u_1, U_2 = u_2, \ldots, U_n = u_n) = P(U_{l+1} = u_1, U_{l+2} = u_2, \ldots, U_{l+n} = u_n)$$
>
> for every n, every shift l and all $(u_1, u_2, \ldots, u_n) \in \mathcal{U}^n$.

- Mean, variance etc. do not change with n.
- Can still have arbitrary time dependence.

Examples

IID sequences: e.g., sequence of fair iid coin tosses

Examples: Stationary time-invariant Markov processes

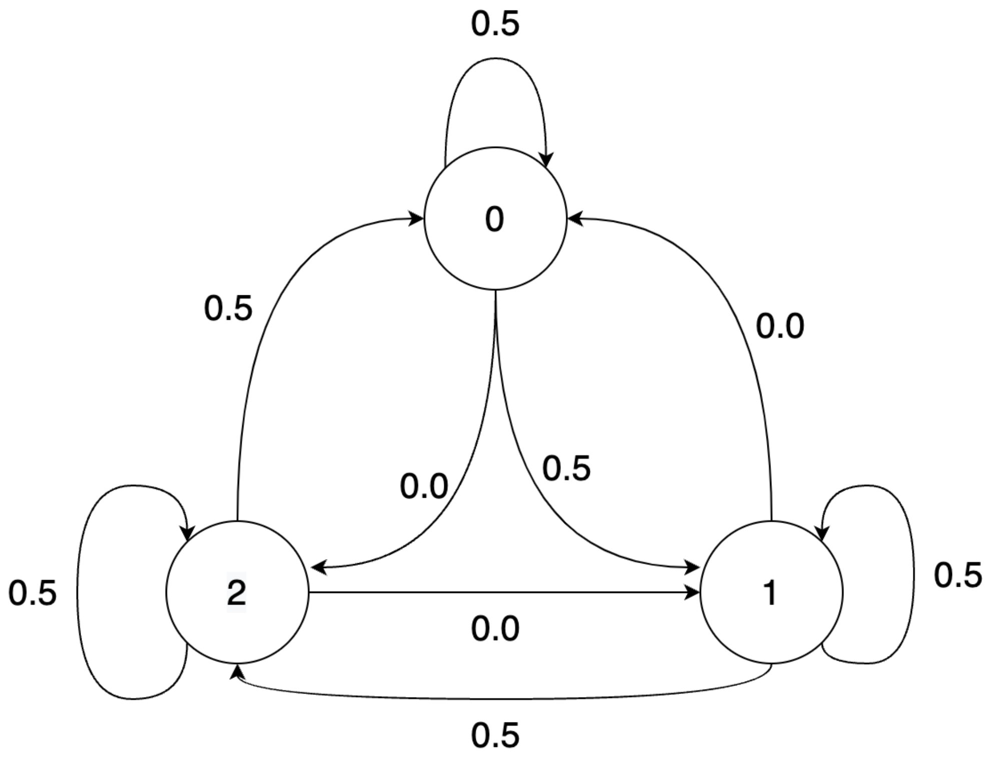
$$U_1 \sim Unif(\{0,1,2\})$$

$$U_{i+1} = (U_i + Z_i) \bmod 3$$

$$Z_i \sim Ber\left(\frac{1}{2}\right)$$

```
Transition matrix
   U_{i+1} 0   1   2
U_i
0         0.5 0.5 0.0
1         0.0 0.5 0.5
2         0.5 0.0 0.5
```

Examples: Stationary time-invariant Markov processes



Question: Can you convert this to an iid sequence?

> All the iid compression work still useful!

kth order Markov source

> **Definition: kth order Markov source**
> A kth order Markov source is defined by the condition
>
> $$P(U_n|U_{n-1}U_{n-2}\ldots) = P(U_n|U_{n-1}U_{n-2}\ldots U_{n-k})$$
>
> for every n. In words, the conditional probability of U_n given the entire past depends only on the past k symbols.

Most practical stationary sources can be approximated well with a finite memory kth order Markov source with higher values of k typically providing a better approximation (with diminishing returns).

Non-example

Arrival times for buses at a bus stop: $U_1, U_2, U_3, U_4, \ldots$

4:16 pm, 4:28 pm, 4:46 pm, 5:02 pm

Question 1: Is this stationary?

Question 2: Can you convert this to a stationary (in fact iid) process?

Information-theoretic quantities for non-iid random variables

Conditional entropy

The conditional entropy of U given V is defined as

$$H(U|V) \triangleq E\left[\log \frac{1}{P(U|V)}\right]$$

Can also write this as

$$\begin{aligned}
H(U|V) &= \sum_{u\in\mathcal{U},v\in\mathcal{V}} P(u,v)\log\frac{1}{P(u|v)} \\
&= \sum_{v\in\mathcal{V}} P(v) \sum_{u\in\mathcal{U}} P(u|v)\log\frac{1}{P(u|v)} \\
&= \sum_{v\in\mathcal{V}} P(v) H(U|V=v)
\end{aligned}$$

Properties of conditional entropy

1. Conditioning reduces entropy: $H(U|V) \leq H(U)$ with equality iff U and V are independent.

Properties of conditional entropy

1. Conditioning reduces entropy: $H(U|V) \leq H(U)$ with equality iff U and V are independent.
2. Chain rule of entropy:

$$H(U, V) = H(U) + H(V|U) = H(V) + H(U|V)$$

Properties of conditional entropy

1. Conditioning reduces entropy: $H(U|V) \leq H(U)$ with equality iff U and V are independent.
2. Chain rule of entropy:

$$H(U,V) = H(U) + H(V|U) = H(V) + H(U|V)$$

3. Joint entropy vs. sum of entropies:

$$H(U,V) \leq H(U) + H(V)$$

with equality holding iff U and V are independent.

Properties of conditional entropy

1. Conditioning reduces entropy: $H(U|V) \leq H(U)$ with equality iff U and V are independent.
2. Chain rule of entropy:

$$H(U,V) = H(U) + H(V|U) = H(V) + H(U|V)$$

3. Joint entropy vs. sum of entropies:

$$H(U,V) \leq H(U) + H(V)$$

with equality holding iff U and V are independent.

Can generalize to conditioning U_{n+1} on $(U_1, U_2, \ldots, U_n)$:

$$H(U_{n+1}|U_1, U_2, \ldots, U_n)$$

Entropy rate

Before we look at examples, let's think about how we can generalize entropy for stationary processes. Some desired criteria:

- works for arbitrarily long dependency so $H(U_{n+1}|U_1, U_2, \ldots, U_n)$ for any finite n won't do
- has *operational* meaning in compression just like entropy
- is well-defined for any stationary process

Entropy rate

Not only one, but two equivalent ways of defining it!

Entropy rate

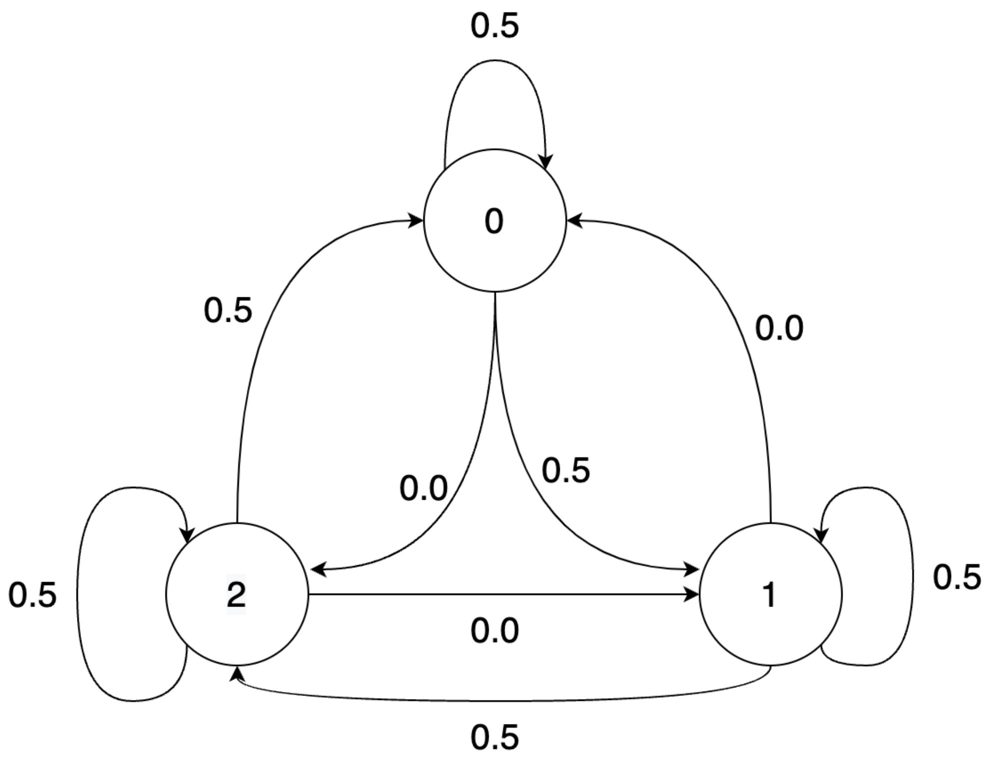
$$H_1(\mathbf{U}) = \lim_{n\to\infty} H(U_{n+1}|U_1, U_2, \ldots, U_n)$$

$$H_2(\mathbf{U}) = \lim_{n\to\infty} \frac{H(U_1, U_2, \ldots, U_n)}{n}$$

> **C&T Thm 4.2.1**
>
> For a stationary stochastic process, the two limits above are equal. We represent the limit as $H(\mathbf{U})$ (entropy rate of the process, also denoted as $H(\mathcal{U})$).

Examples

- Fair coin toss
- Markov example



Example: entropy rate of English text

- Models (estimate probabilities from text):

  (a) 0th-order Markov chain (iid):

  $$H(\mathcal{X}) \approx 4.76 \qquad \text{bits per letter}$$

  (b) 1st order Markov chain:

  $$H(\mathcal{X}) \approx 4.03 \qquad \text{bits per letter}$$

  (c) 4th order Markov chain:

  $$H(\mathcal{X}) \approx 2.8 \qquad \text{bits per letter}$$

- Estimate by asking people to guess the next letter until they get it correct. The *order* of their guesses reflects their estimate of the *order* of their conditional probabilities for the next letter. (Shannon 1952).

  $$H(\mathcal{X}) \approx 1.3 \qquad \text{bits per letter}$$

Source: http://reeves.ee.duke.edu/information_theory/lecture4-Entropy_Rates.pdf

AEP again!

> **Shannon–McMillan–Breiman theorem**
>
> $$-\frac{1}{n}\log_2 P(U_1, U_2, \ldots, U_n) \rightarrow H(\mathbf{U}) \text{ a.s.}$$
>
> under technical conditions (ergodicity).

Takeaway: entropy rate is the best compression you can hope to achieve.

How to achieve the entropy rate?

- Today: we start small, try to achieve kth order entropy $H(U_{k+1}|U_1,\ldots,U_k)$.
- Next week: achieving entropy rate for arbitrary stationary distributions (in theory) and a really performant scheme (in practice). 🤯

Working with known 1st order Markov source

Suppose we know $P(U_2|U_1)$.

How would you go about compressing a block of length n using

$$E\left[\log_2 \frac{1}{P(U_1,\ldots,U_n)}\right] \approx nH(U_2|U_1)$$

bits?

Working with known 1st order Markov source

Idea 1: Use Huffman on blocks of length n.

- Usual concerns: big block size, complexity, etc.
- For non-iid sources, working on independent symbols is just plain suboptimal even discounting the effects of non-dyadic distributions.

Exercise: Compute $H(U_1)$ and $H(U_1, U_2)$ for

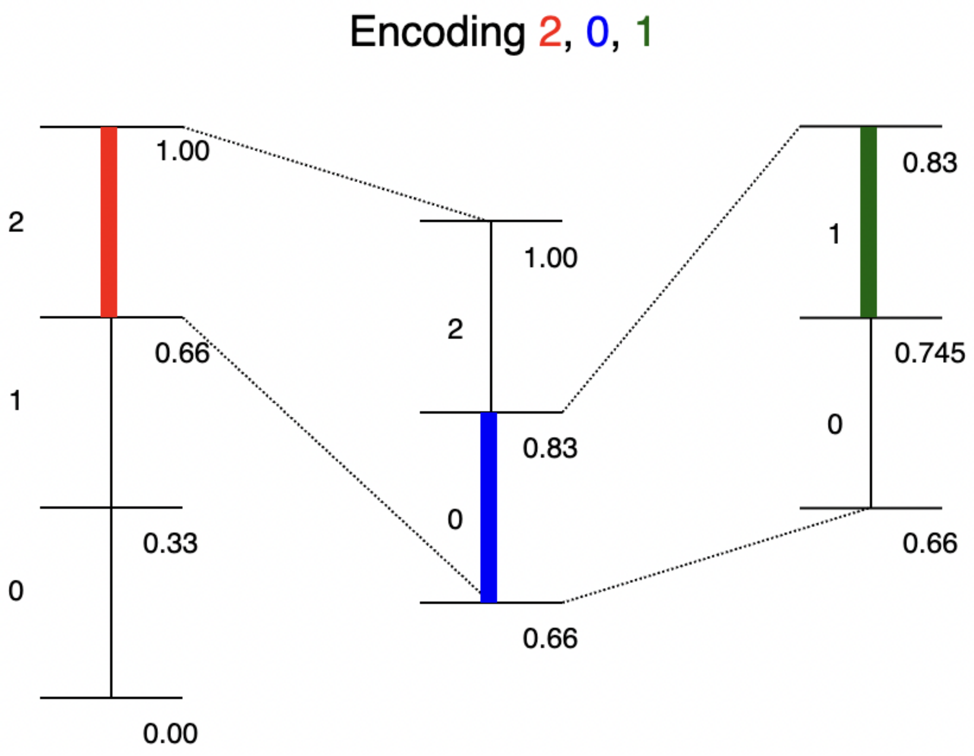
$$U_1 \sim Unif(\{0, 1, 2\})$$

$$U_{i+1} = (U_i + Z_i) \bmod 3$$

$$Z_i \sim Ber\left(\frac{1}{2}\right)$$

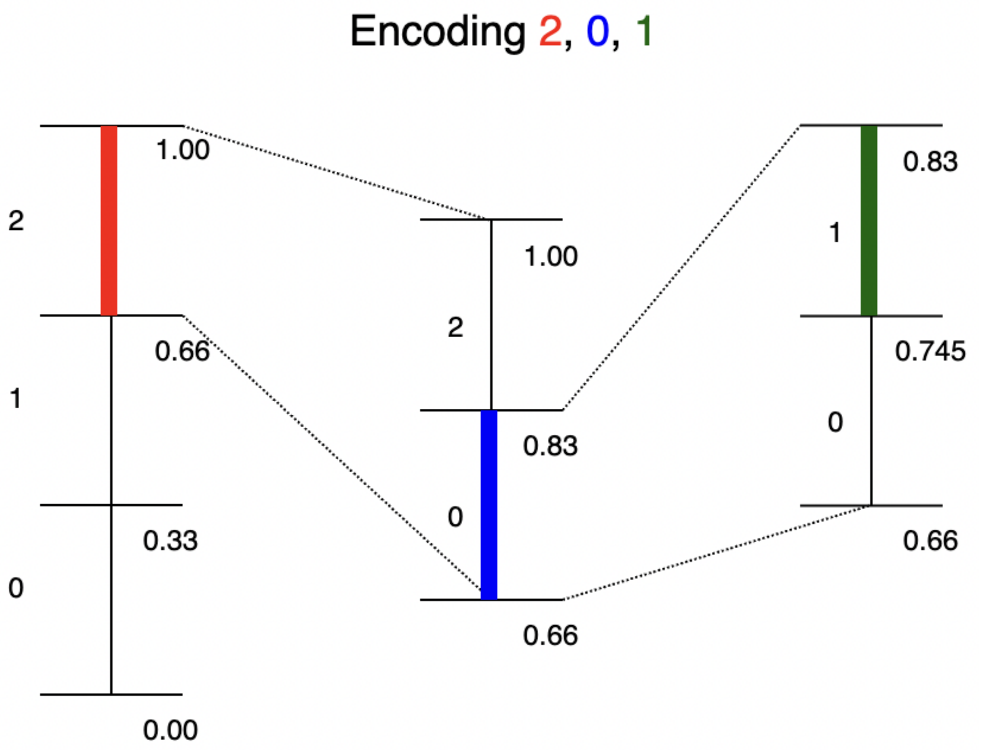
and compare to $H(\mathbf{U})$.

Working with known 1st order Markov source



Question: Can you explain the general idea?

Working with known 1st order Markov source



Question: Can you explain the general idea?

Answer: At every step, split interval by $P(-|u_{i-1})$ [more generally by $P(-|\text{entire past})$].

Arithmetic coding for known 1st order Markov source

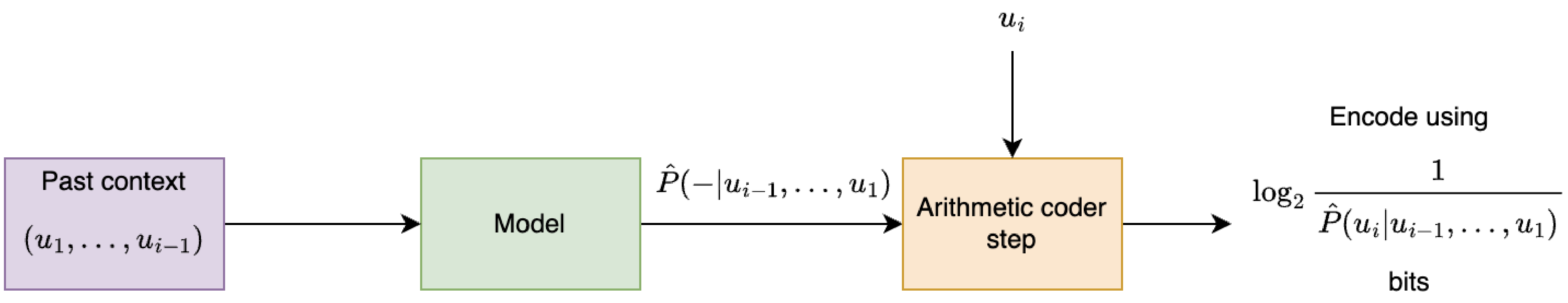
Length of interval after encoding $u_1, u_2, u_3, \ldots, u_n =$
$P(u_1)P(u_2|u_1)\ldots P(u_n|u_{n-1})$

Bits for encoding ~ $\log_2 \frac{1}{P(u_1)P(u_2|u_1)\ldots P(u_n|u_{n-1})}$

Expected bits per symbol

$$\begin{aligned}
&\sim \frac{1}{n} E\left[\log_2 \frac{1}{P(U_1)P(U_2|U_1)\ldots P(U_n|U_{n-1})}\right] \\
&= \frac{1}{n} E\left[\log_2 \frac{1}{P(U_1)}\right] + \frac{1}{n}\sum_{i=2}^{n} E\left[\log_2 \frac{1}{P(U_i|U_{i-1})}\right] \\
&= \frac{1}{n} H(U_1) + \frac{n-1}{n} H(U_2|U_1) \\
&\sim H(U_2|U_1)
\end{aligned}$$

Context-based arithmetic coding

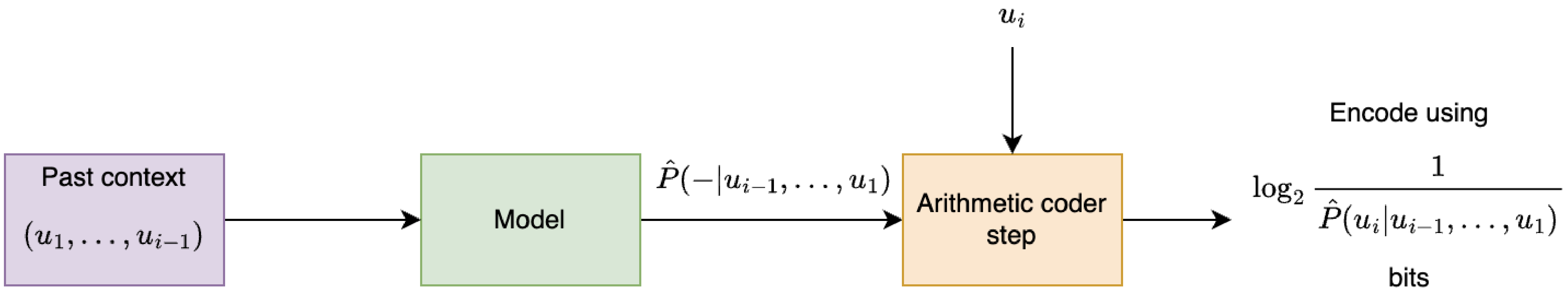


Total bits for encoding:

$$\sum_{i=1}^{n} \log_2 \frac{1}{\hat{P}(u_i|u_1, \ldots, u_{i-1})}$$

Question: How would the decoding work?

Context-based arithmetic coding



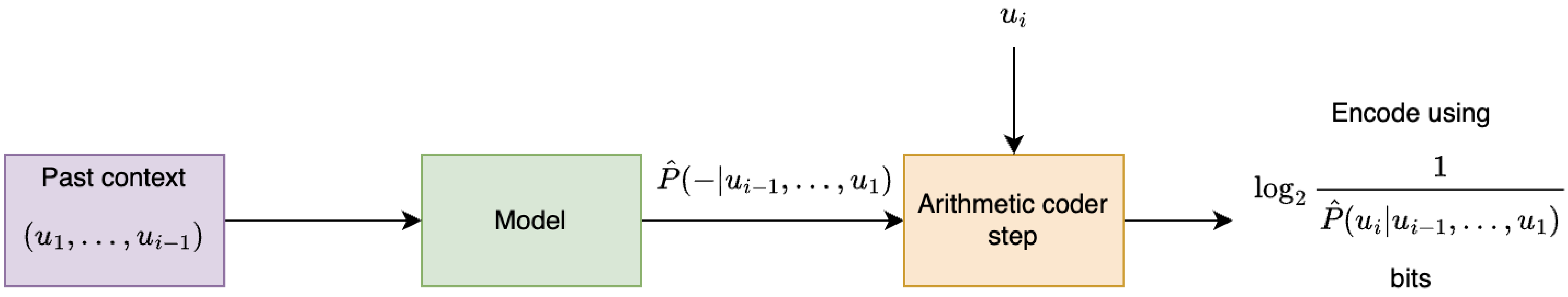
Total bits for encoding:

$$\sum_{i=1}^{n} \log_2 \frac{1}{\hat{P}(u_i|u_1, \ldots, u_{i-1})}$$

Question: How would the decoding work?

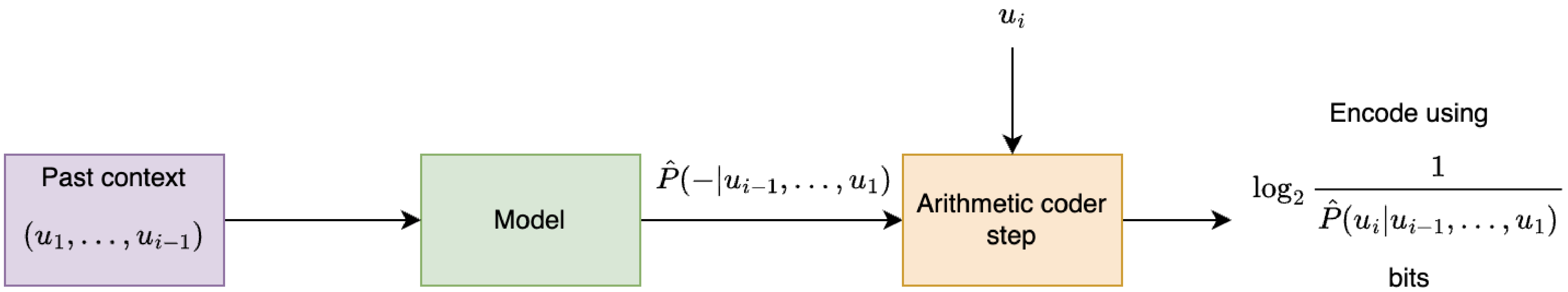
Answer: Decoder uses same model, at step i it has access to $u_1, \ldots, u_{i-1}$ already decoded and so can generate the $\hat{P}$ for the arithmetic coding step!

Context-based arithmetic coding



Question: I don't already have a model. What should I do?

Context-based arithmetic coding



Question: I don't already have a model? What should I do?

Option 1: Two pass: first build ("train") model from data, then encode using it.

Option 2: Adaptive: build ("train") model from data as we see it (more on this shortly).

Two-pass vs. adaptive

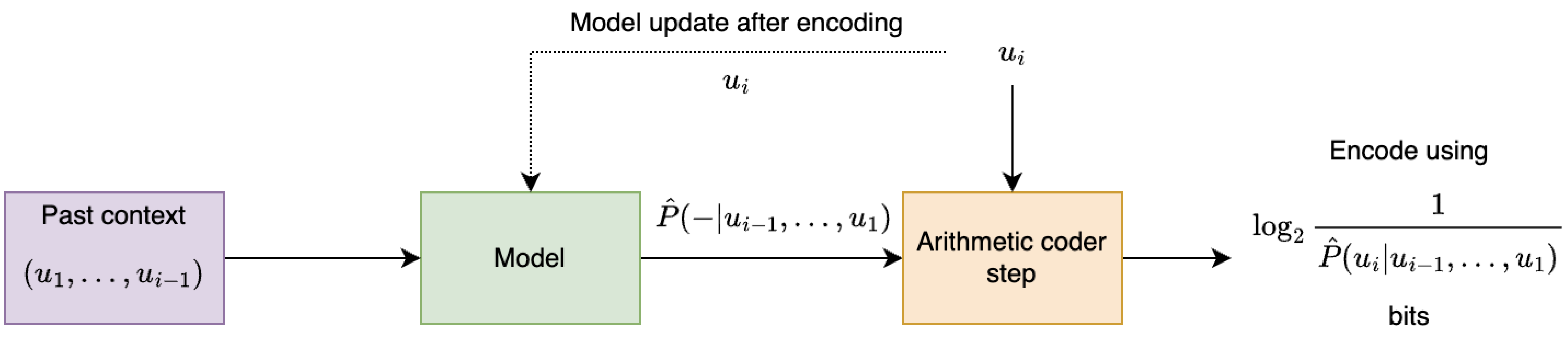
Two-pass approach

✅ learn model from entire data, leading to potentially better compression
✅ more suited for parallelization
❌ need to store model in compressed file
❌ need two passes over data, not suitable for streaming
❌ might not work well with changing statistics

Adaptive approach

✅ no need to store the model
✅ suitable for streaming
❌ adaptively learning model leads to inefficiency for initial samples
✅ works pretty well in practice!

Adaptive context-based arithmetic coding



⚠️ Important for encoder and decoder to share exactly the same model state at every step (including at initialization).

⚠️ Don't go about updating model with u_i before you perform the encoding for u_i.

⚠️ Try not to provide 0 probability to any symbol.

Compression and prediction

Cross-entropy loss for prediction (classes $\mathcal{C}$, predicted probabilities $\hat{P}$, ground truth class: y):

$$\sum_{c \in \mathcal{C}} \mathbf{1}_{y_i = c} \log_2 \frac{1}{\hat{P}(c | y_1, \ldots, y_{i-1})}$$

Loss incurred when ground truth is y_i is $\log_2 \frac{1}{\hat{P}(y_i | y_1, \ldots, y_{i-1})}$

Exactly matches the number of bits used for encoding with arithmetic coding!

Compression and prediction

- Good prediction => Good compression
- Compression = having a good model for the data
- Need not always explicitly model the data

Compression and prediction

- Each compressor induces a predictor!
- Recall relation between code length and induced probability model $p \sim 2^{-l}$
- Generalizes to prediction setting
- Explicitly obtaining the prediction probabilities easier with some compressors than others

Compression and prediction

- Each compressor induces a predictor!
- Recall relation between code length and induced probability model $p \sim 2^{-l}$
- Generalizes to prediction setting
- Explicitly obtaining the prediction probabilities easier with some compressors than others

Prediction models used for compression

kth order adaptive arithmetic coding

```python
def freqs_current(self):
    """Calculate the current freqs. We use the past k symbols to pick out
    the corresponding frequencies for the (k+1)th.
    """
        freqs_given_context = np.ravel(self.freqs_kplus1_tuple[tuple(self.past_k)])
```

```python
def update_model(self, s):
    """function to update the probability model. This basically involves update the count
    for the most recently seen (k+1) tuple.

    Args:
        s (Symbol): the next symbol
    """
    # updates the model based on the new symbol
    # index self.freqs_kplus1_tuple using (past_k, s) [need to map s to index]
    self.freqs_kplus1_tuple[(*self.past_k, s)] += 1

    self.past_k = self.past_k[1:] + [s]]
```

kth order adaptive arithmetic coding

On `sherlock.txt`:

```
>>> with open("sherlock.txt") as f:
>>>     data = f.read()
>>>
>>> data_block = DataBlock(data)
>>> alphabet = list(data_block.get_alphabet())
>>> model_params = (alphabet, order)
>>> encoder = ArithmeticEncoder(AECParams(), model_params, AdaptiveOrderKFreqModel)
>>> encoded_bitarray = encoder.encode_block(data_block)
```

kth order adaptive arithmetic coding

Compressor	bits/char
0th order	4.12
1st order	3.34
2nd order	**2.85**
3rd order	3.09
gzip	2.78
bzip2	**2.06**

kth order adaptive arithmetic coding

Compressor	bits/char
0th order	4.12
1st order	3.34
2nd order	**2.85**
3rd order	3.09
gzip	2.78
bzip2	**2.06**

Question: Why is order 3 doing worse than order 2?

kth order adaptive arithmetic coding

Limitations

- slow, complexity grows exponentially in k
- counts become very sparse for large k, leading to worse performance
- unable to exploit similarities in prediction for *similar* contexts

Some of these can be overcome with smarter modeling as discussed next.

Note: Despite their performance limitations, context based models are still employed as the entropy coding stage after suitably preprocessing the data (LZ, BWT, etc.).

Prediction models used for compression

- kth order adaptive (in SCL): https://github.com/kedartatwawadi/stanford_compression_library/blob/main/compressors/probability_models.py
- Bit-level models
- Context Tree Weighting (CTW)
- Prediction by Partial Matching (PPM)
- Neural net based: NNCP, Tensorflow-compress, DZip
- Ensemble methods: CMIX

These are some of the most powerful compressors around, but often too slow to use in practice!

DeepZip framework

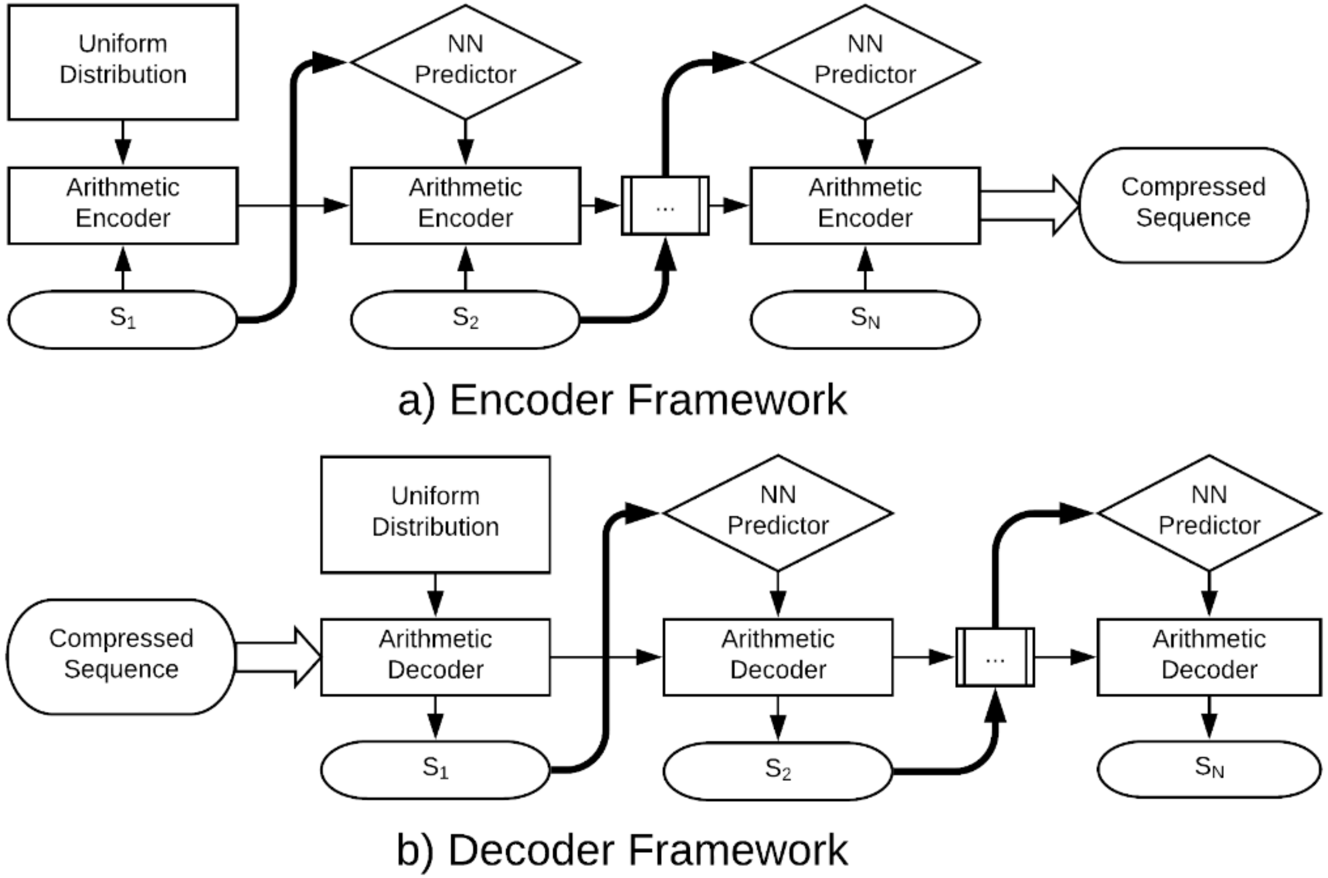
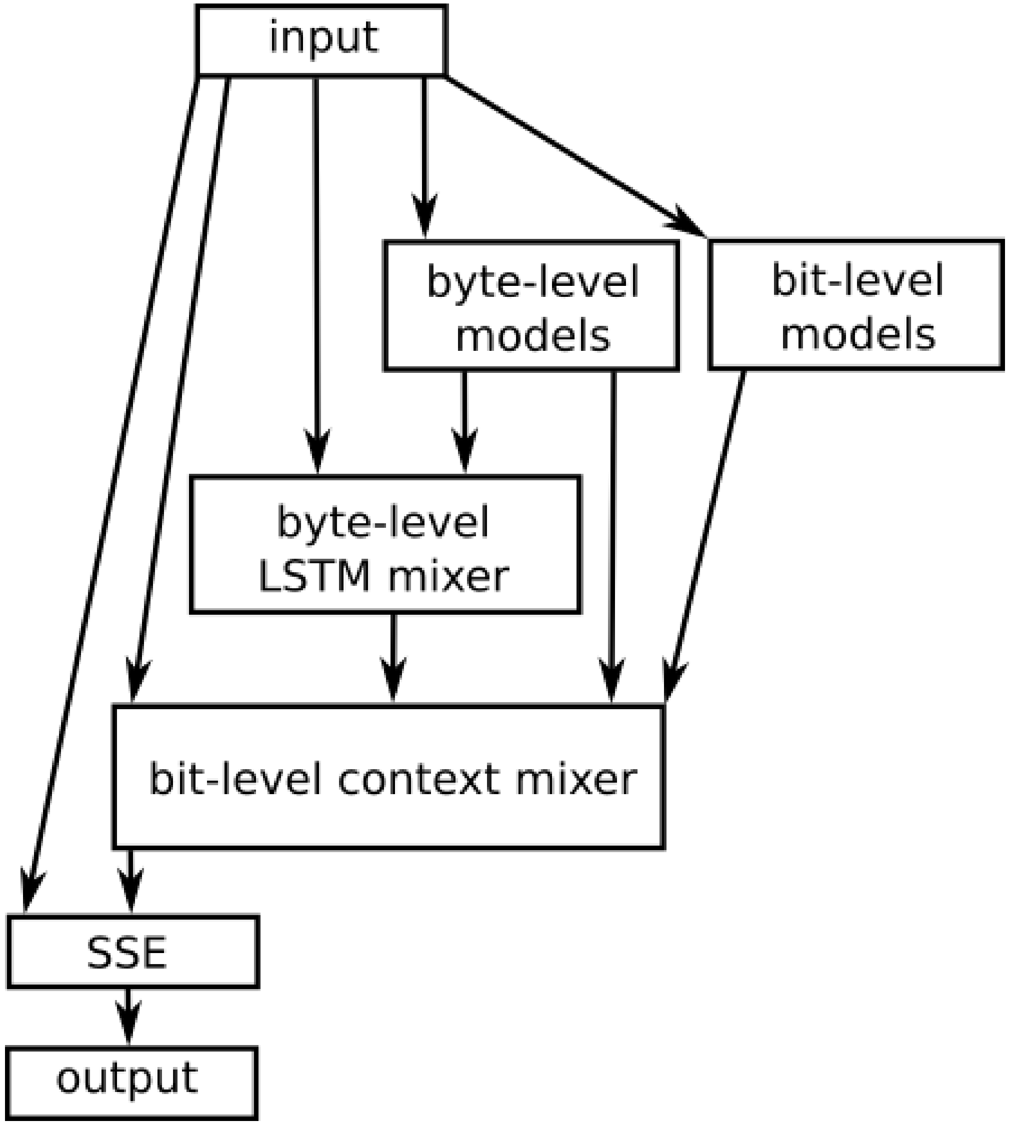
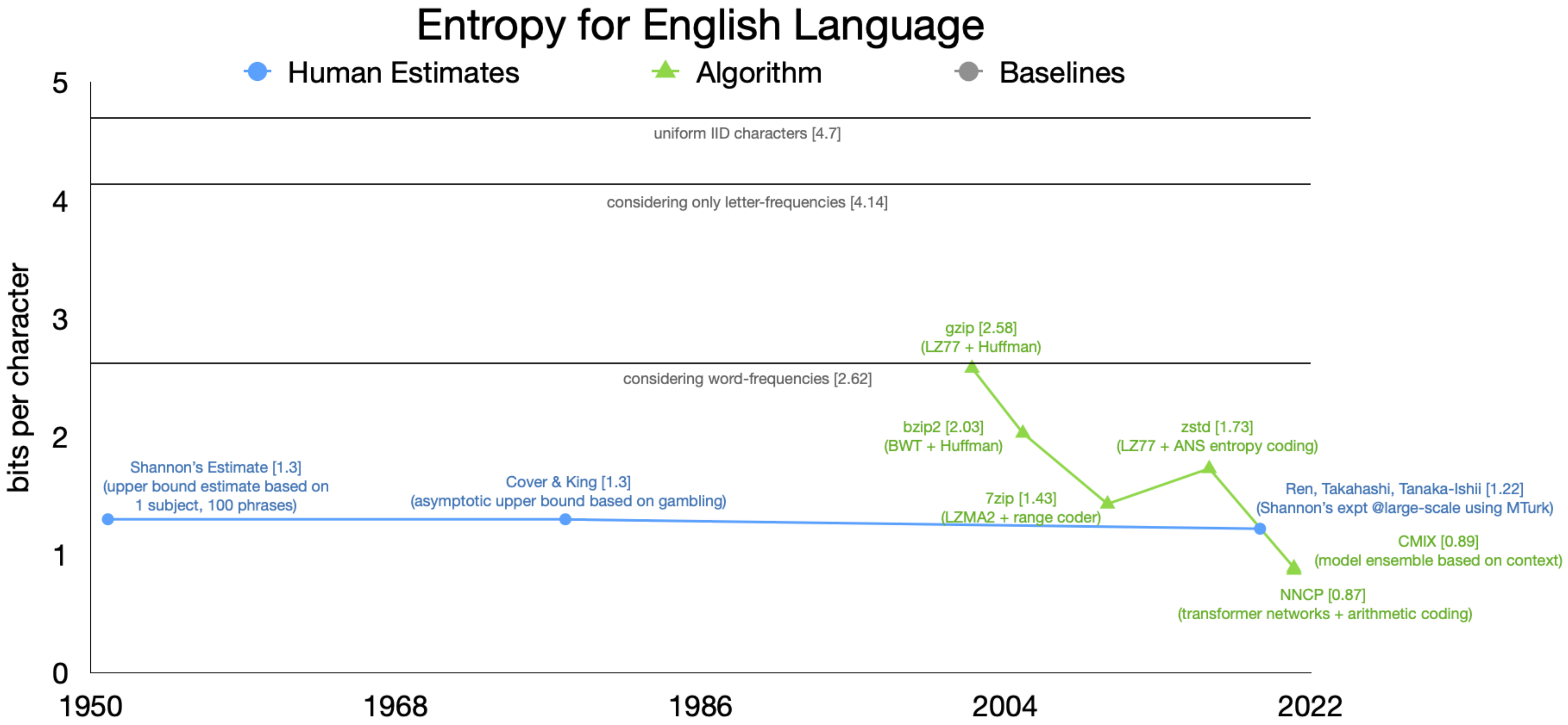


Figure 1: Encoder-Decoder Framework.

CMIX context mixing



Text compression over the years



Next week

- Lempel-Ziv algorithms - the most widely used algorithms in practice!